

# Zero Dispersion and Zero Dissipation Implicit Runge-Kutta Methods for the Numerical Solution of Oscillating IVPs

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## Abstract

In this paper we present two new methods based on an implicit Runge-Kutta method Gauss which is of algebraic order fourth and has two stages: the first one has zero dispersion and the second one has zero dispersion and zero dissipation. The efficiency of these methods is measured while integrating the radial Schrödinger equation and other well known initial value problems.

*Key words:* Runge-Kutta, implicit method, Gauss method, Dispersion, Dissipation, Stability, Initial Value Problems (IVPs), radial Schrödinger equation, resonance problem, energy,

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## 1. Introduction

We consider the radial Schrödinger equation:

$$y''(x) = \left[ \frac{l(l+1)}{x^2} + V(x) - E \right] y(x) \quad (1)$$

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where  $\frac{l(l+1)}{x^2}$  is the centrifugal potential,  $V(x)$  is the potential,  $E$  is the energy and  $W(x) = \frac{l(l+1)}{x^2} + V(x)$  is the effective potential. It is valid that

$$\lim_{v \rightarrow \infty} V(x) = 0$$

and therefore

$$\lim_{v \rightarrow \infty} W(x) = 0.$$

We will study the case of  $E > 0$ .

If we divide  $[0, \infty]$  into small subintervals  $[a_i, b_i]$  so that  $W(x)$  is considered constant with value  $\bar{W}_i$ , then the problem (1) is reduced to the approximation

$$y_i'' = (\bar{W} - E) y_i, \text{ whose solution is}$$

$$y_i(x) = A_i \exp\left(\sqrt{\bar{W} - E}x\right) + B_i \exp\left(-\sqrt{\bar{W} - E}x\right), \quad A_i, B_i \in \mathfrak{R}. \quad (2)$$

This form of Schrödinger equation shows why phase fitting is so important when new methods are constructed. In the next section we will present the most important parts of the theory used.

The structure of the paper is as follows. Firstly in section 2, the basic theory of implicit Runge-Kutta methods is presented. In section 3, the construction of the methods is introduced. In section 4 and 5, the calculation of the algebraic order and the symplecticity respectively of the new methods is given. Then in section 6, the numerical results of the new methods are presented compared to classical RK methods from the literature while integrating well known initial value problems and the radial Schrödinger equation. Finally in section 7 our conclusions are presented.

## 2. Basic Theory

### 2.1. Implicit Method.

The general form of an s-stage implicit Runge-Kutta method used for the computation of the approximate value of  $y_{n+1}(x)$  in Problem (1), when  $y_n(x)$  is known, is given from the following procedure:

$$\begin{aligned} w_i &= f(t_n + c_i h, y_n + h \sum_{j=1}^s a_{ij} w_j) \\ y_{n+1} &= y_n + h \sum_{i=1}^s b_i w_i \end{aligned} \quad (3)$$

when at least one  $a_{ij} \neq 0$  exists with  $i \leq j$ .

An implicit Runge-Kutta method can also be presented using the Butcher table below:

$$\begin{array}{c|cccccc}
 c_1 & a_{11} & a_{12} & \dots & a_{1s-1} & a_{1s} \\
 c_2 & a_{21} & a_{22} & \dots & a_{2s-1} & a_{2s} \\
 c_3 & a_{31} & a_{32} & \dots & a_{3s-1} & a_{3s} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 c_s & a_{s1} & a_{s2} & \dots & a_{s,s-1} & a_{ss} \\
 \hline
 & b_1 & b_2 & \dots & b_{s-1} & b_s
 \end{array} \tag{4}$$

## 2.2. Phase-Lag Analysis

Let A and B,  $s \times s$  matrices, be defined by  $A = (a_{ij})$ ,  $(1 \leq i, j \leq s)$  and  $B = (a_{ij} - b_j)$ ,  $(1 \leq i, j \leq s)$  respectively. When the method (3) is applied to the linear equation

$$y' = qy, \quad q \in C \tag{5}$$

the numerical solution is given by

$$y_{n+1} = P(z)y_n, \quad P(z) = \frac{\det(I - zB)}{\det(I - zA)} \tag{6}$$

and can be written in the form

$$P(z) = K(v) + iL(v), \quad z = hq \tag{7}$$

where  $K(v)$  and  $L(v)$  are functions of  $v$  and  $i = \sqrt{-1}$ .

**Definition 1.** [6] In the implicit s-stage Runge-Kutta method, presented in (4), the quantities

$$\phi(v) = v - \arg(P(iv)), \quad \alpha(v) = 1 - |P(iv)|, \quad v \in \Re \tag{8}$$

are respectively called the phase-lag or dispersion error and the dissipative error. If  $\phi(v) = O(v^{q+1})$  and  $\alpha(v) = O(v^{r+1})$  then the method is said to be of dispersive order  $q$  and dissipative order  $r$ .

### 2.3. Stability

**Definition 2.** [15] The stability function for an implicit Runge-Kutta method is the rational function

$$R(z) = \frac{\det(I - zA + zeb^T)}{\det(I - zA)}, \quad (9)$$

where the vector  $e = (1, \dots, 1)^T$ , and that a method is A-stable if  $|R(z)| \leq 1$ , whenever  $\operatorname{Re}(z) \leq 0$ , where  $\operatorname{Re}(z)$  is the real part of  $z$ .

### 3. Construction of the new Runge-Kutta methods

We consider the implicit Runge-Kutta method of Gauss, which is of algebraic order fourth and has two stages. The coefficients are shown in Table 10.

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad (10)$$

Below we present the construction of the methods.

#### 3.1. Construction of the new method with zero phase lag

We consider all the values of Table 10 except  $b_2$ . By evaluating the phase-lag of this method, defined in Definition 1, and by solving  $\phi(v) = 0$  towards  $b_2$ , the result is:

$$b_2 = \frac{1}{2} \frac{-6v^3 + 6v^3\sqrt{3} + 72v - \tan(v)v^4 + 24\tan(v)v^2}{\tan(v)\sqrt{3}v^4 + 6v^3 + 6v^3\sqrt{3} + \tan(v)v^4\sqrt{3} - 12\tan(v)v^2\sqrt{3} - 144\tan(v)} \frac{-36v^2\tan(v) - 12v^2\tan(v)\sqrt{3} - 72v}{-36v^2\tan(v) - 12v^2\tan(v)\sqrt{3} - 72v} \quad (11)$$

The Taylor series expansion of  $b_2$  is shown below:

$$b_{2_{\text{taylor}}} = \frac{1}{2} + \frac{1}{720}v^4 + \left( \frac{1}{6720} - \frac{1}{8640}\sqrt{3} \right)v^6 + \dots$$

In the last equation we observe that

$$\lim_{v \rightarrow 0} b_{2_{taylor}} = \frac{1}{2}$$

namely when the step-length tends to zero the coefficient of the method Gauss appears.

### 3.2. Construction of the new method with zero phase lag and zero dissipation

We consider all the values of Table 10 except two:  $b_2$  and  $a_{22}$ . An extra equation (apart from the equation of the phase lag) must hold, in order to achieve zero phase-lag and zero dissipation. The two equations are  $\phi(v) = 0$  and  $\alpha(v) = 0$ .

After satisfying the above two equations, by solving towards  $b_2$  and  $a_{22}$ , the result is:

$$b_2 = \frac{1}{6} \cdot \frac{A}{B} \tag{12}$$

where

$$\begin{aligned} A = & -24 \sin(v) v \sqrt{3} + 2 \sin(v) v^3 \sqrt{3} + 12 v^2 \cos(v) \\ & -36 \sin(v) v - 3 \sin(v) v^3 - 144 \cos(v) \\ & + 12 \sqrt{-\frac{-4 v^4 + 2 v^4 \sqrt{3} + 24 v^2 - 24 v^2 \sqrt{3} - 144}{(\cos(v))^2}} \cos(v) \end{aligned}$$

$$B = v \left( 4 v \sqrt{3} \cos(v) + \sin(v) v^2 - 12 \sin(v) \right)$$

and

$$a_{22} = -\frac{1}{12} \cdot \frac{C}{D} \tag{13}$$

with

$$\begin{aligned}
C = & 41472 + 72 v^4 \sqrt{3} + 864 v^2 - 90 v^6 \sqrt{3} + 1152 v^4 + 162 v^6 \\
& + 12096 v^2 \sqrt{3} - 14688 v^2 (\cos(v))^2 - 144 v^4 (\cos(v))^2 \\
& + 41472 (\cos(v))^2 - 120 \sin(v) v^5 \sqrt{3} \cos(v) \\
& - 864 \sin(v) v T_0 \cos(v) - 504 T_0 \sin(v) v^3 (\cos(v))^2 \cos(v) \\
& - v^7 \sin(v) \sqrt{3} \cos(v) + 20736 \sin(v) v \sqrt{3} \cos(v) \\
& - 1872 \sin(v) v^3 \sqrt{3} \cos(v) - 288 T_0 (\cos(v))^2 v^2 \sqrt{3} (\cos(v))^2 \\
& + 48 v^4 T_0 (\cos(v))^2 \sqrt{3} (\cos(v))^2 + 576 v^2 T_0 (\cos(v))^2 (\cos(v))^2 \\
& - 36 v^5 \sin(v) \cos(v) + 10368 \sin(v) v \cos(v) \\
& + 1728 \sin(v) v^3 \cos(v) + 3 v^7 \sin(v) \cos(v) \\
& - 36 v^4 T_0 (\cos(v))^2 + 132 \sin(v) v^3 \sqrt{3} T_0 \cos(v) \\
& + v^5 \sin(v) \sqrt{3} T_0 \cos(v) - 1728 \sin(v) v \sqrt{3} T_0 \cos(v) \\
& + 18 v^6 (\cos(v))^2 - 6912 T_0 (\cos(v))^2 6 v^6 \sqrt{3} (\cos(v))^2 \\
& - 1728 v^2 \sqrt{3} (\cos(v))^2 - 360 v^4 \sqrt{3} (\cos(v))^2
\end{aligned}$$

$$\begin{aligned}
D = & -144 v^4 (\cos(v))^2 - 6 v^6 (\cos(v))^2 + 576 v^2 \sqrt{3} + 864 v^2 \\
& - 144 \sin(v) v^3 \sqrt{3} \cos(v) - 2 v^6 \sqrt{3} - 6 v^5 + 120 v^4 (\cos(v))^3 \sqrt{3} \\
& - 36 v^5 \cos(v) \sin(v) - 1728 v^2 \sqrt{3} (\cos(v))^2 \\
& - 24 \sin(v) v^5 \sqrt{3} \cos(v) - 1152 v^3 \cos(v) \sin(v) \\
& + 3456 \cos(v) \sin(v) v + 96 v^2 \sqrt{3} (\cos(v))^2 - v^7 \sin(v) \sqrt{3} \cos(v) \\
& + 24 T_0 \sin(v) v^3 \cos(v) + 12 v^4 (\cos(v))^2 \\
& + 14 v^6 \sqrt{3} (\cos(v))^2 + 3 v^7 \sin(v) \cos(v) \\
& + v^6 \sin(v) \sqrt{3} \cos(v) - 12 \sin(v) v^4 \sqrt{3} \cos(v) \\
& - 288 \sin(v) \cos(v) - 24 v^3 \sqrt{3} - 864 v (\cos(v))^2
\end{aligned}$$

where

$$T_0 = \sqrt{-\frac{2 v^4 \sqrt{3} - 4 v^4 + 24 v^2 - 24 v^2 \sqrt{3} - 144}{(\cos(v))^2}}$$

The Taylor series expansion of  $b_2$  and  $a_{22}$  are shown below:

$$b_{2_{taylor}} = \frac{1}{2} + \frac{1}{720} v^4 + \frac{1}{10080} \frac{5\sqrt{3}-8}{-3+\sqrt{3}} v^6 + \dots$$

$$a_{22_{taylor}} = \frac{1}{4} + \frac{1}{2160} \frac{5\sqrt{3}-9}{-2+\sqrt{3}} v^4 - \frac{1}{181440} \frac{220\sqrt{3}-381}{(-2+\sqrt{3})^2} v^6 + \dots$$

In the last equations we observe that the limits when  $v \rightarrow 0$  are equal to the corresponding coefficients of the Gauss method.

#### 4. Algebraic order of the new methods

The following 8 equations must be satisfied so that the new method maintains the fourth algebraic order of the corresponding classical method presented in Table 10. The number of stages is symbolized by  $s$ , where  $s = 2$ .

**1st Alg. Order (1 equation)**

$$\sum_{i=1}^s b_i = 1 \quad (14)$$

**2st Alg. Order (2 equations)**

$$\sum_{i=1}^s b_i c_i = \frac{1}{2} \quad (15)$$

**3st Alg. Order (4 equations)**

$$\begin{aligned} \sum_{i=1}^s b_i c_i^2 &= \frac{1}{3} \\ \sum_{i,j=1}^s b_i a_{ij} c_j &= \frac{1}{6} \end{aligned} \quad (16)$$

**4st Alg. Order (8 equations)**

$$\begin{aligned} \sum_{i=1}^s b_i c_i^3 &= \frac{1}{4} \\ \sum_{i,j=1}^s b_i c_i a_{ij} c_j &= \frac{1}{8} \\ \sum_{i,j=1}^s b_i a_{ij} c_j^2 &= \frac{1}{12} \\ \sum_{i,j,k=1}^s b_i a_{ij} a_{jk} c_k &= \frac{1}{24} \end{aligned} \quad (17)$$

#### 4.1. Remainders for the first method (algebraic conditions)

We present the remainders of the eight equations, that is the difference of the right part minus the left part, for the first method:

$$\begin{aligned}
rem_1 &= \frac{1}{720} v^4 + \left( \frac{1}{6720} - \frac{1}{8640} \sqrt{3} \right) v^6 + \dots \\
rem_2 &= \left( \frac{1}{1440} + \frac{\sqrt{3}}{4320} \right) v^4 + \left( \frac{1}{60480} - \frac{\sqrt{3}}{30240} \right) v^6 + \dots \\
rem_3 &= \left( \frac{1}{2160} + \frac{\sqrt{3}}{4320} \right) v^4 - \left( \frac{1}{120960} - \frac{\sqrt{3}}{72576} \right) v^6 + \dots \\
rem_4 &= \left( \frac{1}{4320} + \frac{\sqrt{3}}{8640} \right) v^4 - \left( \frac{1}{241920} - \frac{\sqrt{3}}{145152} \right) v^6 + \dots \\
rem_5 &= \left( \frac{1}{2880} + \frac{\sqrt{3}}{5184} \right) v^4 - \left( \frac{1}{90720} - \frac{\sqrt{3}}{120960} \right) v^6 + \dots \\
rem_6 &= \left( \frac{1}{5760} + \frac{\sqrt{3}}{10368} \right) v^4 - \left( \frac{1}{181440} - \frac{\sqrt{3}}{241920} \right) v^6 + \dots \\
rem_7 &= \left( \frac{1}{8640} + \frac{\sqrt{3}}{12960} \right) v^4 - \left( \frac{1}{145152} - \frac{\sqrt{3}}{725760} \right) v^6 + \dots \\
rem_8 &= \left( \frac{1}{17280} + \frac{\sqrt{3}}{25920} \right) v^4 - \left( \frac{1}{290304} - \frac{\sqrt{3}}{1451520} \right) v^6 + \dots
\end{aligned} \tag{18}$$

We see that the eight equations are held, when  $h \rightarrow 0 \Rightarrow v \rightarrow 0$ . This means that the new method maintains the algebraic order of the corresponding classical method.

#### 4.2. Remainders for the second method (algebraic conditions)

Now we present the remainders of the equations for the second method:

$$\begin{aligned}
rem_1 &= \frac{1}{720} v^4 + \frac{1}{10080} \frac{-8+5\sqrt{3}}{\sqrt{3}-3} v^6 + \dots \\
rem_2 &= \left( \frac{1}{1440} + \frac{1}{4320} \sqrt{3} \right) v^4 + \frac{1}{60480} \frac{(-8+5\sqrt{3})(3+\sqrt{3})}{\sqrt{3}-3} v^6 + \dots \\
rem_3 &= \frac{1}{25920} (3+\sqrt{3})^2 v^4 + \frac{1}{362880} \frac{(-8+5\sqrt{3})(3+\sqrt{3})^2}{\sqrt{3}-3} v^6 + \dots \\
rem_4 &= -\frac{1}{8640} \frac{11\sqrt{3}-21}{(\sqrt{3}-3)(-2+\sqrt{3})} v^4 + \frac{1}{181440} \frac{137\sqrt{3}-237}{(\sqrt{3}-3)(-2+\sqrt{3})^2} v^6 + \dots \\
rem_5 &= \frac{1}{155520} (3+\sqrt{3})^3 v^4 + \frac{1}{2177280} \frac{(-8+5\sqrt{3})(3+\sqrt{3})^3}{\sqrt{3}-3} v^6 + \dots \\
rem_6 &= -\frac{1}{51840} \frac{(3+\sqrt{3})(11\sqrt{3}-21)}{(\sqrt{3}-3)(-2+\sqrt{3})} v^4 + \frac{1}{1088640} \frac{(3+\sqrt{3})(137\sqrt{3}-237)}{(\sqrt{3}-3)(-2+\sqrt{3})^2} v^6 + \dots \\
rem_7 &= -\frac{1}{8640} (-2+\sqrt{3})^{-1} v^4 + \frac{1}{725760} \frac{-53+31\sqrt{3}}{(\sqrt{3}-3)(-2+\sqrt{3})^2} v^6 + \dots \\
rem_8 &= -\frac{1}{17280} \frac{3385\sqrt{3}-5863}{(\sqrt{3}-3)(-2+\sqrt{3})^5} v^4 + \frac{1}{290304} \frac{28121\sqrt{3}-48707}{(\sqrt{3}-3)(-2+\sqrt{3})^6} v^6 + \dots
\end{aligned} \tag{19}$$

We see that for  $v = 0$  the eight equations are held for this method too. Thus the new method has also fourth algebraic order.



## 5. Symplecticity of the new methods

**Theorem** The Runge-Kutta method (3)-(4) is symplectic when the following equalities are satisfied

$$b_i a_{ij} + b_j a_{ji} = b_i b_j, \quad 1 \leq i, j \leq s. \quad (20)$$

As a classical example we mention the Gauss methods as symplectic Runge-Kutta methods. It should be noted that symplectic Runge-Kutta methods are always implicit.

Thus according to the above theorem, the three equations must be satisfied so that the symplecticity of the new methods will be maintained.

$$b_1 a_{11} + b_1 a_{11} = b_1 b_1 \quad (21)$$

$$b_2 a_{22} + b_2 a_{22} = b_2 b_2 \quad (22)$$

$$b_1 a_{12} + b_2 a_{21} = b_1 b_2 \quad (23)$$

### 5.1. Remainders for the first method (symplecticity conditions)

We present the remainder of the three equations, that is the difference of the right part minus the left part, for the first method:

$$\begin{aligned} rem_1 &= 0 \\ rem_2 &= -\frac{1}{1440} v^4 + \left(-\frac{1}{13440} + \frac{1}{17280} \sqrt{3}\right) v^6 + \dots \\ rem_3 &= \left(-\frac{1}{2880} + \frac{1}{4320} \sqrt{3}\right) v^4 + \left(-\frac{23}{241920} + \frac{13}{241920} \sqrt{3}\right) v^6 + \dots \end{aligned} \quad (24)$$

We see that for the three equations are held, when  $h \rightarrow 0 \Rightarrow v \rightarrow 0$ . That means that the new method maintains the symplecticity of the corresponding classical method.

### 5.2. Remainders for the second method (symplecticity conditions)

Now we present the remainders of the equations for the second method:

$$\begin{aligned} rem_1 &= 0 \\ rem_2 &= \frac{1}{720} \frac{-45+26\sqrt{3}}{(-2+\sqrt{3})(-3+\sqrt{3})^2} v^4 + \frac{1}{10080} \frac{545\sqrt{3}-944}{(-2+\sqrt{3})^2(-3+\sqrt{3})^3} v^6 + \dots \\ rem_3 &= -\frac{1}{2880} \frac{-5+3\sqrt{3}}{-3+\sqrt{3}} v^4 - \frac{1}{120960} \frac{-54+31\sqrt{3}}{-3+\sqrt{3}} v^6 + \dots \end{aligned} \quad (25)$$

We see that for  $v = 0$  the three equations are held for this method too. Thus the new method is also symplectic.

## 6. Numerical Results

### 6.1. The methods

In order to measure the efficiency of the methods constructed in this paper we compare them to some already known methods, presenting the results of the best six.

**I.** Method **G2-PL-D** constructed in this paper, where G2-PL-D means the method Gauss two-stages, fourth-order with zero phase-lag and zero dissipation.

**II.** Method **G2-PL** constructed in this paper, where G2-PL means the method Gauss two-stage, fourth-order with zero phase-lag.

**III.** Method **G2**: The classical two-stages and fourth-order Gauss method (see [16]).

**IV.** Method **SDIRK(3,6,3)**: The Singly Diagonally-Implicit Runge-Kutta method of J. M. Franco, I. Gomez, L. Randez, is third-stage, third algebraic order, sixth dispersive order and third dissipative order (see [25]).

**V.** Method **Radau I**: The classical third order Radau method (see [15]).

**VI.** Method **Lobatto IIIC**: The classical fourth order Lobatto method (see [15]).

### 6.2. The Problems

#### 6.2.1. Inverse Resonance Problem

The efficiency of the two new constructed methods will be measured through the integration of problem (1) with  $l = 0$  at the interval  $[0, 15]$  using the well known Woods-Saxon potential

$$V(x) = \frac{u_0}{1+q} + \frac{u_1 q}{(1+q)^2}, \quad q = \exp\left(\frac{x-x_0}{a}\right), \quad (26)$$

where  $u_0 = -50$ ,  $a = 0.6$ ,  $x_0 = 7$  and  $u_1 = -\frac{u_0}{a}$   
and with boundary condition  $y(0) = 0$ .

The potential  $V(x)$  decays more quickly than  $\frac{l(l+1)}{x^2}$ , so for large  $x$  (asymptotic region) the Schrödinger equation (1) becomes

$$y''(x) = \left[ \frac{l(l+1)}{x^2} + V(x) - E \right] y(x) \quad (27)$$

The last equation has two linearly independent solutions  $kxj_l(kx)$  and  $kxn_l(kx)$ , where  $j_l$  and  $n_l$  are the spherical Bessel and Neumann functions. When  $x \rightarrow \infty$  the solution takes the asymptotic form

$$y(x) \approx Akxj_l(kx) - Bkxn_l(kx) \quad (28)$$

$$\approx D [\sin(kx - \pi l/2) + \tan(\delta_l) \cos(kx - \pi l/2)] \quad (29)$$

where  $\delta_l$  is called scattering phase shift and it is given by the following expression:

$$\tan(\delta_l) = \frac{y(x_i)S(x_{i+1}) - y(x_{i+1})S(x_i)}{y(x_{i+1})C(x_i) - y(x_i)C(x_{i+1})} \quad (30)$$

where  $S(x) = kxj_l(kx)$ ,  $C(x) = kxn_l(kx)$  and  $x_i < x_{i+1}$  and both belong to the asymptotic region. Given the energy we approximate the phase shift, the accurate value of which is  $\pi/2$  for the above problem. We will use two values for the energy: 989.701916 and 341.495874. As for the frequency  $w$  we will use the suggestion of Ixaru and Rizea in [2] and [4].:

$$w = \begin{cases} \sqrt{E-50}, & \text{if } x \in [0, 6.5] \\ \sqrt{E}, & \text{else } x \in [6.5, 15] \end{cases} \quad (31)$$

In Figure 1 we use  $E = 989.701916$  and in Figure 2 we use  $E = 341.495874$ .

### 6.2.2. Inhomogeneous Equation

$y'' = -100y + 99\sin(t)$ , with  $y(0)=1, y'(0)=11, t \in [0, 1000\pi]$ . Theoretical solution:  $y(x) = \sin(t) + \sin(10t) + \cos(10t)$ .

Estimated frequency:  $w=10$ .

### 6.2.3. Duffing Equation

$y'' = -y - y^3 + 0.002 \cos(1.01t)$ , with  $y(0)=0.200426728067, y'(0)=0, t \in [0, 1000\pi]$ . Theoretical solution:  $y(x) = 0.200179477536 \cos(1.01t) + 2.46946143 * 10^{-4} \cos(3.03t) + 3.04014 * 10^{-7} \cos(5.05t) + 3.74 * 10^{-10} \cos(7.07t) +$

...

Estimated frequency:  $w=1$ .

#### 6.2.4. Nonlinear Problem

$y'' = -100y + \sin(y)$ , with  $y(0)=0$ ,  $y'(0)=1$ ,  $t \in [0, 20\pi]$ . The theoretical solution is not known, but we use  $y(20\pi) = 3.92823991 \times 10^{-4}$  and  $w=10$  as frequency of this problem.

### 7. Conclusions

In the following figures we present the accuracy of the tested methods in connection with  $\log_{10}$  of the total steps  $\times$  stages of the methods. Firstly, in figure 1 (Resonance Problem) we use  $E=989.701916$  and we observe that the second method, which has zero phase-lag and zero dissipation, has such the same accuracy with the first method, which has zero phase-lag, but much better accuracy than the other methods (G2, SDIRK(3,6,3), Radau I, Lobatto IIIC), while in figure 2 (Resonance Problem) we use  $E=341.495874$  and we also see that the second method, which has zero phase-lag and zero dissipation, has the same accuracy with the first method, which has zero phase-lag, but better accuracy than the other methods (G2, SDIRK(3,6,3), Radau I, Lobatto IIIC). The conclusion from the above is that the difference in efficiency is higher when using higher energy. Secondly in figure 3 (Inhomogeneous Equation) it can be observed that the second method developed here with zero phase-lag and zero dissipation has way higher efficiency than the first method developed here with zero phase-lag, which has far higher efficiency than the other methods G2, SDIRK(3,6,3), Radau I, Lobatto IIIC. Thirdly, in figure 4 (Duffing Equation) the two new methods have almost the same efficiency but much better than the other four methods. Finally in figure 5 (Nonlinear Equation) the two new methods have almost the same efficiency but much better than the other four methods. This shows the importance of zero phase-lag and zero dissipation in this type of problems.

The stability region for the three methods for  $v = \omega h = 50$  are shown in Figure 6. The stability regions of the methods include the exterior of the curves. Note that the curves of all the three methods include the left half-plane.

## Figures of the Numerical Results

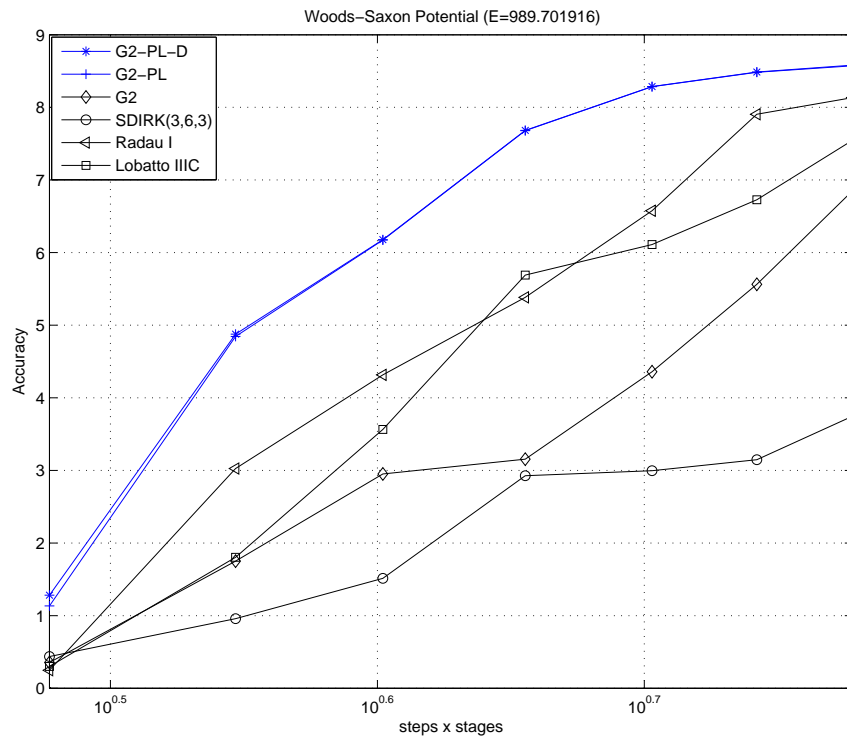


Figure 1: Resonance Problem using  $E=989.701916$

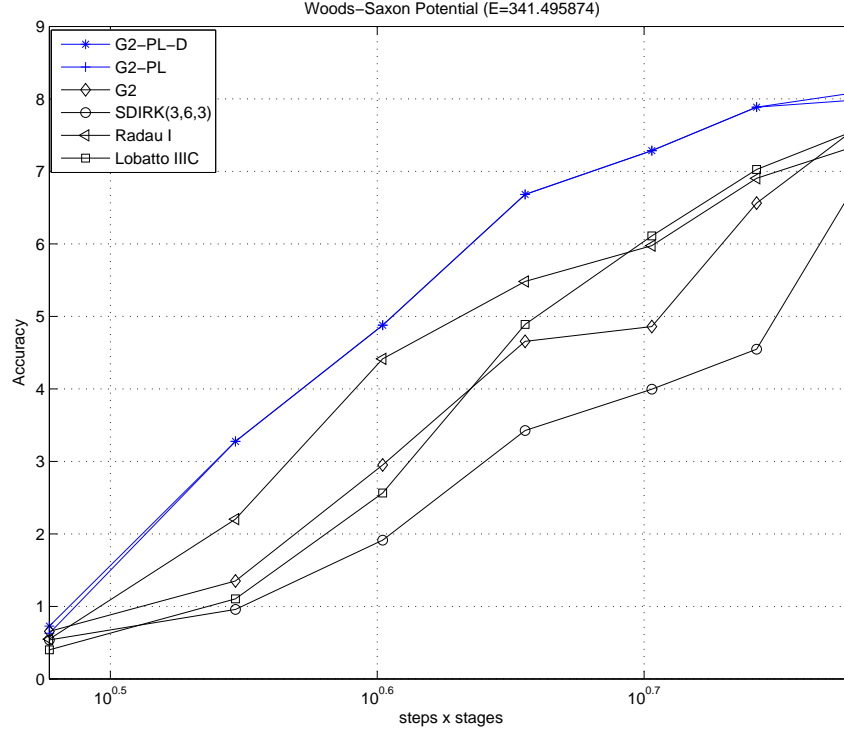


Figure 2: Resonance Problem using  $E=341.495874$

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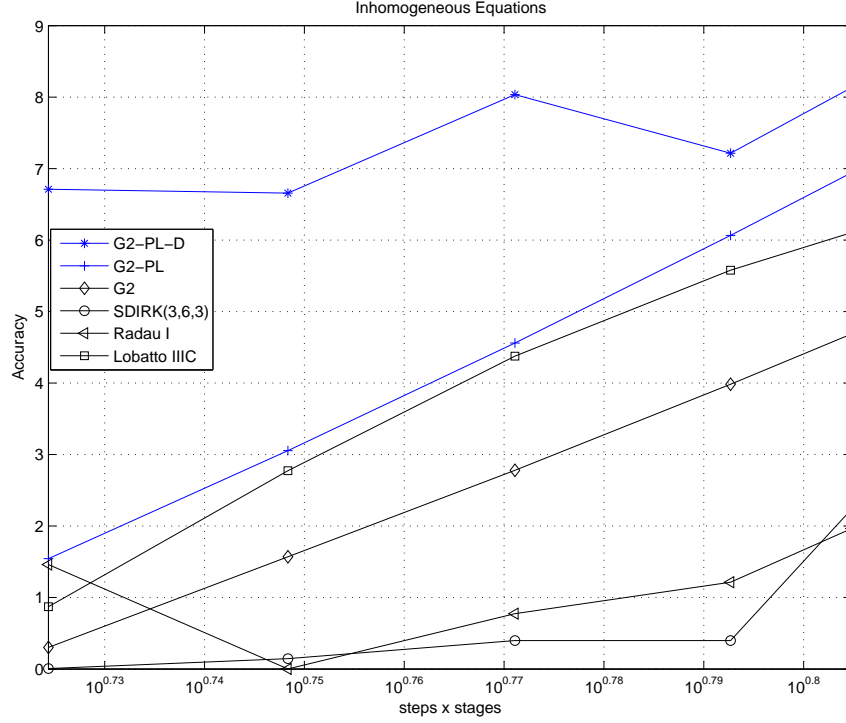


Figure 3: Inhomogeneous Equation

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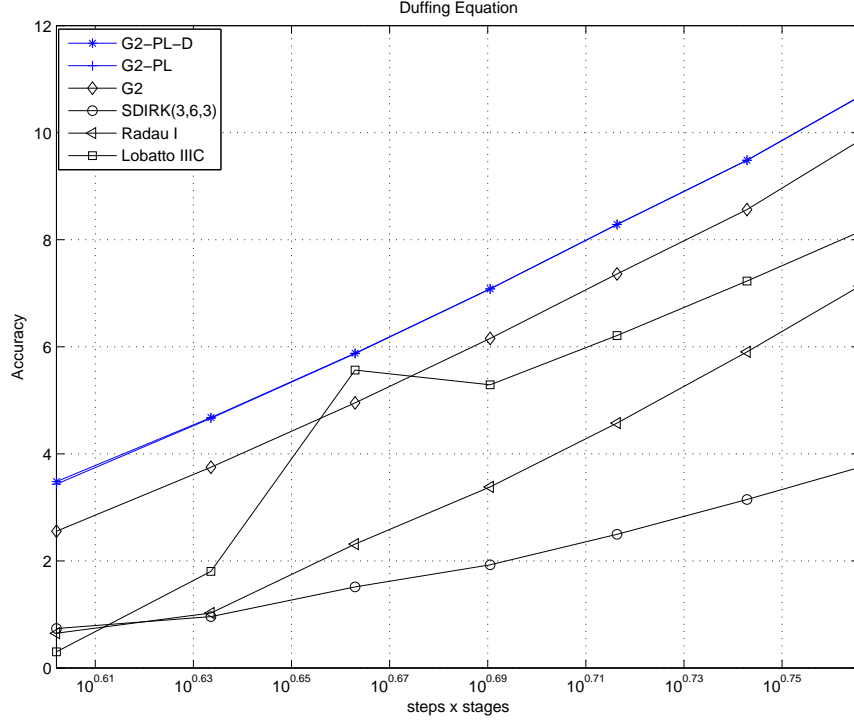


Figure 4: Duffing Equation

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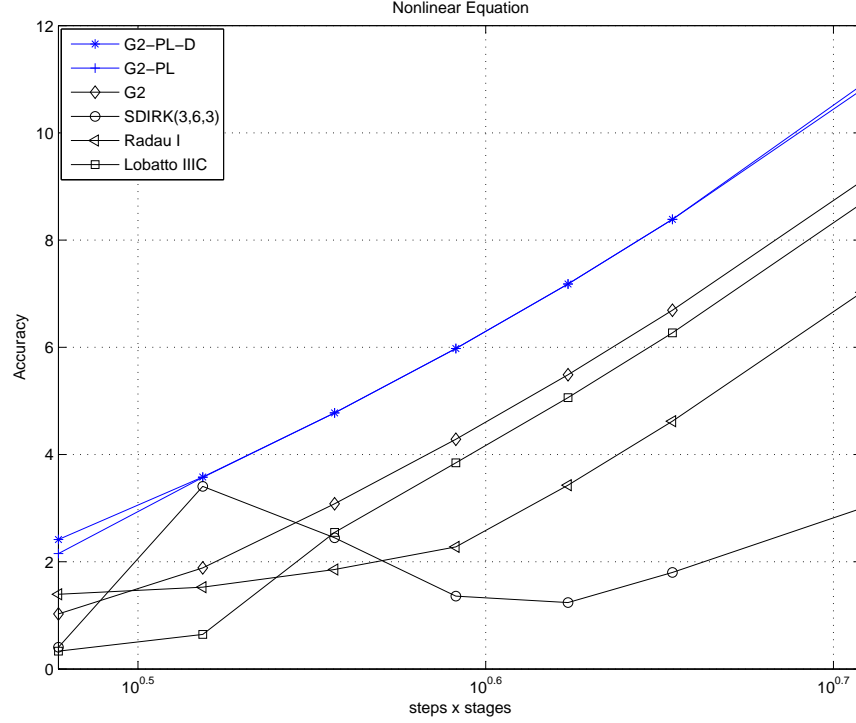


Figure 5: Nonlinear Problem

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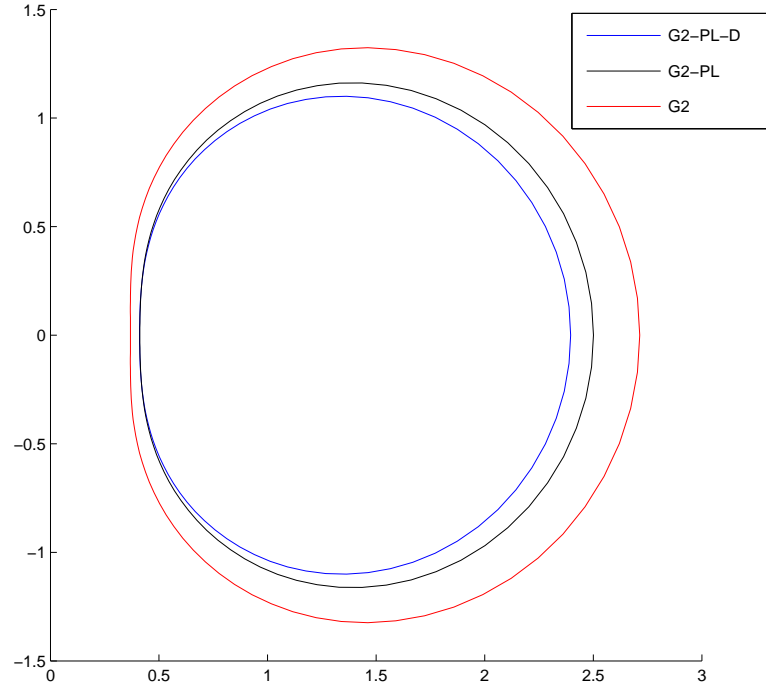


Figure 6: Stability region for the three implicit Runge-Kutta methods

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